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LETTER TO THE EDITOR

A q-deformed harmonic oscillator in a finite-dimensional Hilbert space

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Abstract. A q-deformed harmonic oscillator (q-HO) in a finite-dimensional Hilbert space (FDHS) is presented. Some basic characteristics of the q-HO in the FDHS is discussed. It is found that the quantum algebra su_q (1.1) can be realized in terms of the q-HO in the FDHS on the root of unit.

Recently, the Pegg–Barnett quantum phase theory [1–3] has attracted a lot of attention. It has been applied to various problems in quantum optics (see [4] and references therein). The core of the Pegg–Barrnett thoery is that it introduces a Hermitian phase operator which is defined in a finite-dimensional Hilbert space (FDHS). A harmonic oscillator (HO) defined in the FDHS is proved to play an important role in the construction of the Hermitian phase operator. Because of the finiteness of the Hilbert space, the creation and annihilation and number operators of the HO in the FDHS cannot for the Heisenberg–Weyl algebra (HWA), which is the dynamical algebra of the usual HO in an infinite-dimensional Hilbert space (IDHS). it was found that an HO in an FDHS has some remarkable properties which an HO in an IDHS cannot share [5].

On the other hand, a q-deformed harmonic oscillator (q-HO) [6–8] is known to play an important role in the representation theory and physical applications for quantum groups. In fact, it is a well known fact that the (q-)HO is one of the most fundamental and important objects in quantum and mathematical physics. The purpose of this letter is to present a q-HO in an FDHS and discuss some problems related to the q-HO in the FDHS.

Let us consider an $(s+1)$-dimensional Fock space $\Sigma_{s+1} \equiv \{|0\rangle, |1\rangle, |2\rangle, \ldots, |s\rangle; s$, a finite positive integer$\}$ which is an orthonormal and complete Hilbert space

$$\langle n|m\rangle = \delta_{n,m}. \qquad \sum_{n=0}^{s} |n\rangle\langle n| = 1. \tag{1}$$

Using the projection operator $|n\rangle\langle m|$ in the FDHS, we can define the creation and annihilation operators of a q-HO in the FDHS $\geqslant_{s+1}$

$$a_q = \sum_{n=0}^{s} \sqrt{[n]}\,|n-1\rangle\langle n| \qquad a_q^+ = \sum_{n=0}^{s} \sqrt{[n]}\,|n\rangle\langle n-1| \tag{2}$$

and the number operator

$$N = \sum_{n=0}^{s} n|n-1\rangle\langle n| \tag{3}$$

where the symbol $[x] \equiv (q^x - q^{-x})/(q-q^{-1})$.

It follows from the above definitions that

$$a_q^+ a_q = [N] \qquad a_q^+ a_q = q[N] + q^{-N} - [s+1]\,|s\rangle\langle s|. \tag{4}$$

Making use of (2)–(4), we know that the annihilation and number operators act on Σ_{s+1} as usual

$$a_q|n\rangle = \sqrt{[n]}\,|n-1\rangle \qquad a_q|0\rangle = 0 \tag{5}$$

$$N|n\rangle = n|n\rangle \qquad |n\rangle \in \sum_{s+1} \tag{6}$$

but the action of the creation operator is modified when acting on the highest state $|s\rangle$

$$a_q^+|n\rangle = \sqrt{[n+1]}\,|n+1\rangle \qquad a_q^+|s\rangle = 0 \qquad |n\rangle \in \sum_{s+1}. \tag{7}$$

From (2), (5) and (7), one can find the nilpotency of the creation and annihilation operators of the q-HO in the FDHS

$$a_q^n = 0 \qquad (a_q^+)^n = 0 \qquad (n > s). \tag{8}$$

It is now easy to show that the operators a_q, a_q^+ and N of the q-HO in the FDHS satisfy the following communication relations (CR),

$$a_q a_q^+ - q a_q^+ a_q = q^{-N} - [s+1]\,|s\rangle\langle s| \tag{9}$$

$$[N, a_q] = -a_q \qquad [N, a_q^+] = a_q^+ \tag{10}$$

which means that these operators a_q, a_q^+ and N cannot form the q-HWA for general values of s and q.

Because of the invariance of the q-number $[x]$ under the exchange $q \leftrightarrow q^{-1}$, equation (9) can also be rewritten as

$$a_q a_q^+ - q^{-1} a_q^+ a_q = q^N - [s+1]\,|s\rangle\langle s|. \tag{11}$$

The additional term on the right-hand side of (9) and (11) orginates from the finiteness of the dimension of the FDHS, it makes the q-HO in the FDHS non-trivially different from the q-HO in the IDHS.

These CR of the q-HO in the FDHS (9) and (10) contain remarkable mathematical properties which have potential applications in physics and mathematical physics. We here discuss the following three cases.

(i) When the deformation parameter q tends to 1, equations (9) and (10) reduce to

$$aa^+ - a^+a = 1 - (s+1)|s\rangle\langle s| \tag{12}$$

$$[N, a] = -a \qquad [N, a^+] = a^+ \tag{13}$$

where we have used

$$\lim_{q\to 1} [x] = x$$

and omitted the indexes of the operators. These are the CR of a HO defined in the FDHS Σ_{s+1} which play a fundamental role in the Pegg–Barnett quantum phase theory [1–3].

(ii) When the deformation parameter q takes the root of the unit in the following way

$$q = \exp\left(\frac{2\pi i}{s+1}\right)$$

equations (9) and (10) become

$$a_q a_q^+ - q a_q^+ a_q = q^{-N} \tag{14}$$

$$[N, a_q] = -a_q \qquad [N, a_q^+] = a_q^+ \tag{15}$$

where we have used $[s+1] = 0$ for $q = \exp(2\pi i/(s+1))$. These are the CR of the usual q-HO in the IDHS [6–8]. They show that the operators a_q, a_q^+ and N of the q-HO in the FDHS can form the q-HWA when $q = \exp(2\pi i/(s+1)$. This q-HO in the FDHS on the root of unit can also be used to construct a Hermitian phase operator [9]. When the dimension of the FDHS approaches infinity, i.e. $q \to 1$, (14) and (15) reduce to the CR of the usual HWA:

$$[a, a^+] = 1 \qquad [N, a] = -a \qquad [n, a^+] = a^+. \tag{16}$$

The q-HO in the FDHS on the root of unity $q = \exp(2\pi i/(s+1)$ can be related to the HO in the FDHS through the following deforming maps:

$$a_q = \left(\frac{[N+1]}{N+1}\right)^{1/2} a \qquad a_q^+ = a^+ \left(\frac{[N+1]}{N+1}\right)^{1/2} \tag{17}$$

where a_q, a_q^+ and N satisfy (14) and (15) while a and a^+ obey (12) and (13).

It is interesting to note that the quantum algebra $su_q(1,1)$ can be realized in terms of the q-HO in the FDHS on the root of unit $q = \exp(2\pi i/(s+1))$. To see this, we introduce the operators

$$J_+ = (q+q^{-1})^{-1} a_q^{+2} \qquad J_- = (q+q^{-1})^{-1} a_q^2 \qquad J_3 = -\tfrac{1}{2}(N+\tfrac{1}{2}). \tag{18}$$

With the help of equations (14) and (15), we can prove that the above operators obey the following CR:

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -[2J_3]_{q^2} \tag{19}$$

where $[x]_{q^2}=(q^{2x}-q^{-2x})/(q^2-q^{-2})$. These are just the structure relations of $su_q(1,1)$. Thus, (17) gives rise to a realization of the quantum algebra $\mathrm{su}_q(1,1)$. Apparently, the CR in (19) contract to those of the su(1, 1) algegra in the limit $s\to\infty$, i.e. $q\to 1$.

(iii) When $s=1$, this is a two-state system. In this case, we define the operators

$$K_+=a_q^+\qquad K_-=a_q\qquad K_3=N-\tfrac{1}{2}. \tag{20}$$

From the definitions (2) and (3), they can be rewritten as

$$K_+=|1\rangle\langle 0|\qquad K_-=|0\rangle\langle 1|\qquad K_3=|1\rangle\langle 1|-\tfrac{1}{2}. \tag{21}$$

Now it is easy to check that these operators satisfy the CR of the $su(2)$ algebra

$$[K_3,K_\pm]=\pm K_\pm\qquad [K_+,K_-]=2K_3. \tag{22}$$

So far we have concentrated on mathematical aspects of the q-HO in the FDHS. We now turn to those concerning the physics implied by the q-HO in the FDHS. In order to do so, we introduce the position and momentum operators related to the creation and annihilaton operators as follows:

$$X_q(s)\equiv\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_q^+ + a_q)\qquad P_q(s)\equiv \mathrm{i}(\tfrac{1}{2}m\hbar\omega)^{1/2}(a_q^+ - a_q). \tag{23}$$

Using these operators, we define the Hamiltonian of the q-HO in the FDHS to be

$$H_q(s)=\frac{P_q^2(s)}{2m}+\tfrac{1}{2}m\omega^2X_q^2(s) \tag{24}$$

which can be rewritten as

$$H_q(s)=\tfrac{1}{2}\hbar\omega(a_qa_q^+ + a_qa_q^+) \tag{25}$$

where we have used (23). This Hamiltonian is diagonal on the states $|n\rangle\in\Sigma_{s+1}$ and has the eigenvalues

$$E_n(s)=\tfrac{1}{2}\hbar\omega\{[n]+[n+1]+[s+1]\delta_{n,s}\}. \tag{26}$$

From the above it can be seen that when $n<s$, the energy-level structure of the q-HO in the FDHS is the same as that of the usual q-HO in the IDHS [1, 10]. However the q-HO in the FDHS has the finite highest energy level $E_{n-s}(n)=\frac{1}{2}\hbar\omega\{2[s+1]+[s]\}$ for any finite value of s, which approaches the highest level of the usual q-HO in the limit $s\to\infty$.

In conclusion, we have proposed a q-HO in the FDHS and studied some basic characteristics of the q-HO in the FDHS. It has been shown that the q-HO in the FDHS has rich mathematical properties which may have some applications in physics and mathematical physics. We have also realized the quantum algebra $su_q(1,1)$ in terms of the q-HO in the FDHS on the root of unit.

References

[1] Pegg D T and Barnett S M 1988 *Europhys. Lett.* **6** 483
[2] Barnett S M and Pegg D T 1989 *J. Mod. Opt.* **36** 7
[3] Pegg D T and Barnett S M 1989 *Phys. Rev.* A **39** 1665
[4] Buzeck V, Wilson-Gordon A D and Knight P L 1992 *Phys. Rev.* A **45** 3289

[5] Le-Man Kuang, Fa-Bo Wang and Yan-Guo Zhou 1993 *J. Phys. A: Math. Gen.* (in press)
[6] Biedenharn L C 1989 *J. Phys. A: Math Gen.* **22** L873
[7] Sun C P and Fu H C 1989 *J. Phys. A: Math Gen.* **22** L983
[8] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[9] Ellinas D 1992 *Phys. Rev.* A **45** 3358
[10] Buzeck V 1991 *J. Mod. Opt.* **38** 801